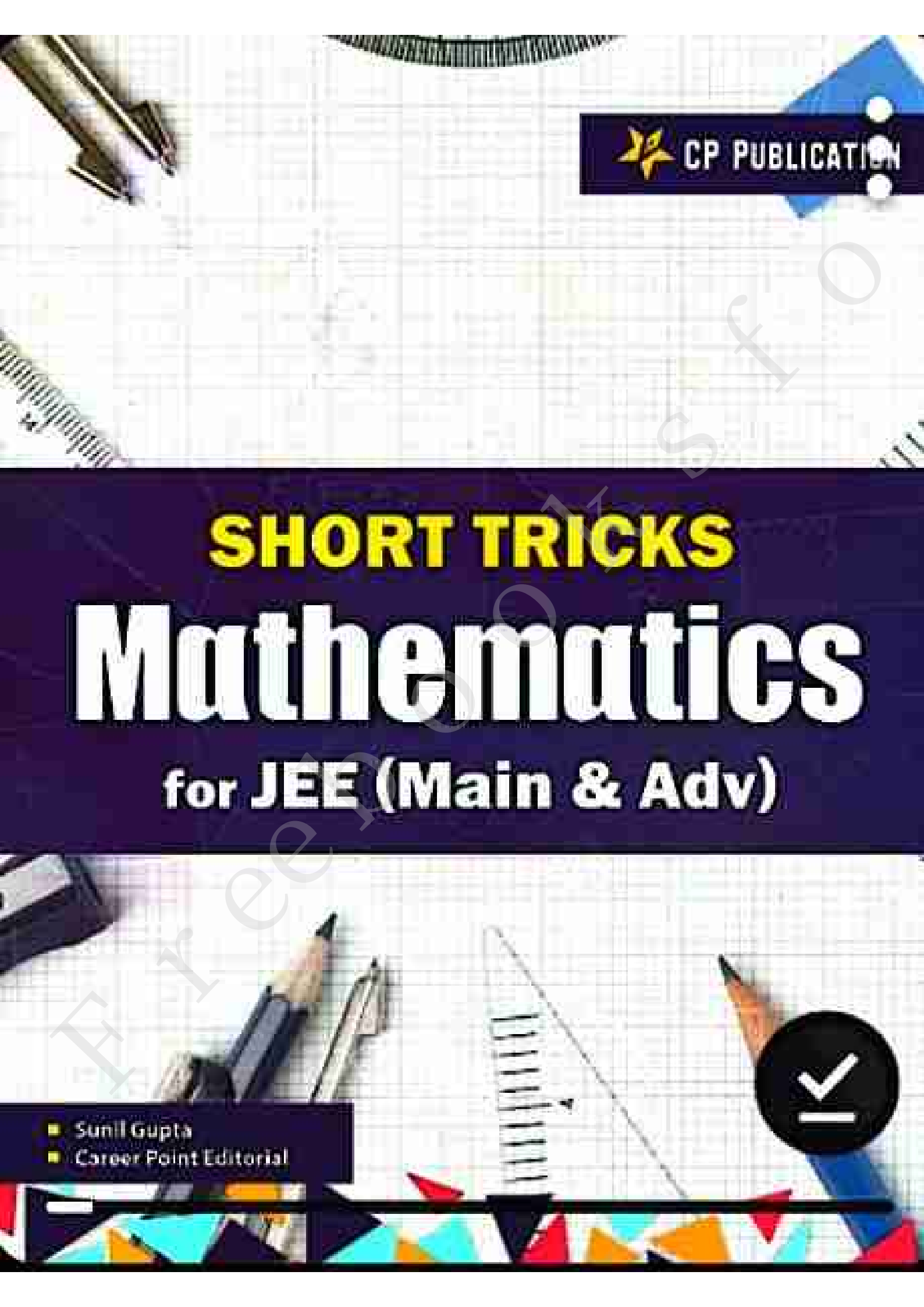
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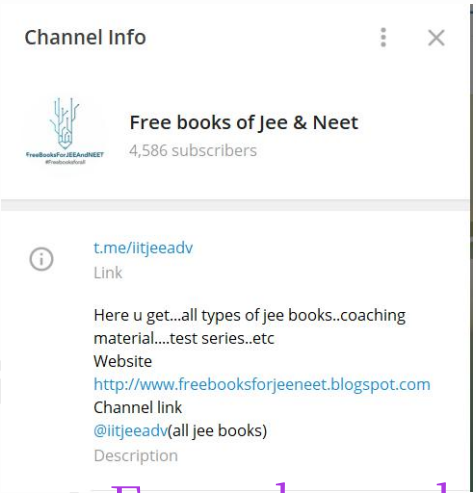

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CONTENTS

	Chapter	Page No.
1.	QUADRATIC EQUATION	01
2.	PROGRESSION	15
3.	COMPLEX NUMBER	35
4.	BINOMIAL THEOREM	51
5.	PERMUTATION & COMBINATION	68
6.	DETERMINANT	83
7.	TRIGONOMETRIC RATIO	110
8.	PROPERTIES OF TRIANGLE	134
9.	INVERSE TRIGONOMETRIC FUNCTION	146
10.	POINT & STRAIGHT LINE	159
11.	CIRCLE	184
12.	CONIC SECTIONS	197
13.	VECTOR	227
14.	FUNCTION	244
15.	LIMIT	280
16.	DIFFERENTIATION	288
17.	APPLICATION OF DERIVATIVE	295

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3

Complex Number

KEY CONCEPTS

1. Real Number System

(i) Natural Number (N) : $N = \{1, 2, 3, ...\}$

(ii) Whole Number (W) : $W = \{0, 1, 2, ...\} = \{N\} + \{0\}$

(iii) Integers (Z or I) : Z or $I = \{.....-3, -2, -1, 0, 1, 2, 3, ...\}$

(iv) Rational Numbers (Q) : The numbers which are in the form of p/q (Where $p, q \in I, q \neq 0$)

(v) Irrational Numbers (Q^c) : The numbers which are not rational i.e. which can not be expressed in p/q form or whose decimal part is non terminating non repeating but which may represent magnitude of physical quantities. e.g. $\sqrt{2}$, $5^{1/3}$, π, e,.....etc.

(vi) Real Numbers (R) : The set of Rational and Irrational Number is called as set of Real Numbers i.e. $N \subset W \subset Z \subset Q \subset R$

2. Imaginary Number

Square root of a negative real number is an imaginary number, while solving equation $x^2 + 1 = 0$ we get $x = \pm\sqrt{-1}$ which is imaginary. So the quantity $\sqrt{-1}$ is denoted by 'i' called 'iota' thus $i = \sqrt{-1}$

3. Integral Power of Iota

$i = \sqrt{-1}$ so $i^2 = -1$; $i^3 = -i$ and $i^4 = 1$

Hence $i^{4n+1} = i$; $i^{4n-2} = -1$; $i^{4n+3} = -i$; i^{4n} or $i^{4n+4} = 1$

4. Complex Number

A number of the form $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$ is called a complex number where x is called as real part and y is called imaginary part of complex number and they are expressed as

$Re(z) = x$, $Im(z) = y$,

Here if $x = 0$ the complex number is purely Imaginary and if $y = 0$ the complex number is purely Real.

Note :

- Inequalities in complex number are not defined because 'i' is neither positive, zero nor negative so $4 + 3i < 1 + 2i$ or $i < 0$ or $i > 0$ is meaning less.
- If two complex numbers are equal, then their real and imaginary parts are separately equal. Thus if $a + ib = c + id \Rightarrow a = c$ and $b = d$
  so if $z = 0 \Rightarrow x + iy = 0 \Rightarrow x = 0$ and $y = 0$
- The complex number 0 is purely real and purely imaginary both.

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5. Algebra of Complex Number

If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be any two complex number then

(i) Addition $(a + ib) + (c + id) = (a + c) + i(b + d)$

(ii) Subtraction $(a + ib) - (c + id) = (a - c) + i(b - d)$

(iii) Multiplication $(a + ib)(c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc)$

(iv) Division $\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$

(when at least one of c and d is non zero)

$$= \frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$$

Note : Additive inverse of z is $-z$ and multiplicative inverse of z is $\frac{1}{z}$.

6. Conjugate of a Complex Number :

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part and it is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$

7. Properties of Conjugate Complex Number

Let $z = a + ib$ and $\bar{z} = a - ib$ then

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2a = 2\,\text{Re}(z)$ = purely real

(iii) $z - \bar{z} = 2ib = 2i\,\text{Im}(z)$ = purely imaginary

(iv) $z\bar{z} = a^2 + b^2 = |z|^2$

(v) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(vi) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(vii) $\overline{re^{i\theta}} = re^{-i\theta}$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(ix) $\overline{z^n} = (\bar{z})^n$

(x) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

(xi) $z + \bar{z} = 0$ or $z = -\bar{z} \Rightarrow z = 0$ or z is purely imaginary

(xii) $z = \bar{z} \Rightarrow$ z is purely real

8. Modulus of a Complex Number

If $z = x + iy$ then modulus of is denoted by $|z|$. Thus

$z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$

9. Properties of Modulus of a Complex Number

(i) $|z| \geq 0$

(ii) $-|z| \leq \text{Re}(z) \leq |z|$

(iii) $-|z| \leq \text{Im}(z) \leq |z|$

(iv) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(v) $z\bar{z} = |z|^2$

(vi) $z^{-1} = \frac{\bar{z}}{|z|^2}$

(vii) $|z_1 z_2| = |z_1|\,|z_2|$

(viii) $|z^n| = |z|^n$

(ix) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, where $|z_2| \neq 0$

(ix) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1\bar{z}_2)$

(x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

(xi) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2|\cos(\theta_1 - \theta_2)$

where θ_1 and θ_2 are the argument of z_1 and z_2

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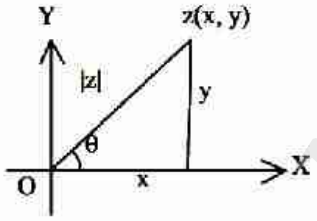
(xii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.

(xiii) $|z_1 \pm z_2| \le |z_1| + |z_2|$ (triangle inequality)

(xiv) $|z_1 \pm z_2| \ge ||z_1| - |z_2||$ (triangle inequality)

10. Amplitude or Argument of Complex Number :

The amplitude or argument of a complex number z is the inclination of the directed line segment representing z, with real axis.



If $z = x + iy$ then amp $(z) = \tan^{-1}\left(\frac{y}{x}\right)$

Note :

- Principle value of any complex number lies between $-\pi < \theta \le \pi$.
- Principal value of argument.
  - (i) If complex number lies in first quadrant then arg $(z) = \theta$.
  - (ii) If complex number lies in second quadrant then arg $(z) = \pi - \theta$.
  - (iii) If complex number lies in third quadrant then arg $(z) = -(\pi - \theta)$.
  - (iv) If complex number lies in fourth quadrant then arg $(z) = -\theta$.

11. Properties of Argument of a Complex Number

(i) amp (any real positive number) = 0

(ii) amp (any real negative number) = π

(iii) amp $(z - \bar{z}) = \pm \frac{\pi}{2}$

(iv) amp $(z_1 z_2)$ = amp(z_1) + amp(z_2) + $2k\pi \ \forall\ k \in I$

(v) amp $\left(\frac{z_1}{z_2}\right)$ = amp (z_1) – amp (z_2) + $2k\pi \ \forall\ k \in I$

(vi) amp $(\bar{z})$ = – amp (z) = amp $\left(\frac{1}{z}\right)$

(vii) amp $(-z)$ = amp $(z) \pm \pi$

(viii) amp $(z^n) = n$ amp $(z) + 2k\pi$ (k = 0, 1 or –1)

(ix) amp (z) + amp $(\bar{z}) = 2k\pi$

$\forall$ proper value of k must be chosen so that R.H.S. lies in $(-\pi, \pi]$.

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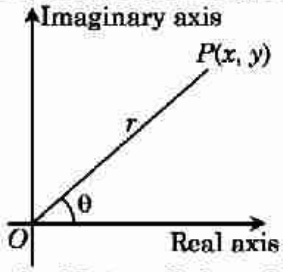
12. Representation of a Complex Number in various forms

(i) Cartesian Form :

Every complex number $z = x + iy$ can be represented by a point on the Cartesian plane known as Complex plane (Argand diagram) by the ordered pair (x, y).



Length OP is called modulus of the complex number denoted by $|z|$ and θ is called the argument or amplitude.

$$|z| = \sqrt{x^2 + y^2} \ \& \ \theta = \tan^{-1}\frac{y}{x}$$

(ii) Trigonometric/Polar Form :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$;

(iii) Eulerian Form :

$z = re^{i\theta}$; $|z| = r$; $\arg z = \theta$; $\bar{z} = re^{-i\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

13. Square Root of a Complex Number

The square root of $z = a + ib$ is

$$\sqrt{a+ib} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right] \text{ for } b > 0 \text{ and}$$

$$= \pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right] \text{ for } b < 0.$$

14. Logarithm of a Complex Number

Let $z = x + iy$ be a complex number and in polar form of z is $re^{i\theta}$, then

$\log(x + iy) = \log(re^{i\theta}) = \log(r) + i\theta$

$$= \log\left(\sqrt{x^2 + y^2}\right) + i\tan^{-1}\frac{y}{x}$$

or $\log z = \log(|z|) + i \text{ amp } (z)$

In general, $z = re^{i(\theta + 2n\pi)}$

$\log z = \log|z| + i\arg z + 2n\pi i$

15. De- Moivre's Theorem

A simplest formula for calculating powers of complex numbers in the form of $\cos\theta$ and $\sin\theta$ is known as De-Moivre's theorem,.

If $n \in I$ (set of integers), then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ and if $n \in Q$ (set of rational numbers), then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Note :

(i) $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)\right]$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots (\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$

(iii) $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos n\theta$

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16 Cube Root of Unity

Cube root of unity are 1, ω, ω^2, where $\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2} = e^{i2\pi/3}$ and $\omega^2 = \frac{-1-i\sqrt{3}}{2}$

$\arg(\omega) = \frac{2\pi}{3}$ and $\arg(\omega^2) = \frac{4\pi}{3}$

Properties of Cube Roots of Unity :

(i) $\omega^3 = 1$ or $\omega^{3r} = 1$

(ii) $1 + \omega + \omega^2 = 0$

(iii) Cube roots of unity lie on the unit circle $|z| = 1$ and divide its circumference into 3 equal parts.

(iv) It always forms an equilateral triangle.

(vi) Cube roots of -1 are $-1, -\omega, -\omega^2$

Important Identities

(a) $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

(b) $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

(c) $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$

(d) $x^2 - xy + y^2 = (x + \omega y)(x + y\omega^2)$

(e) $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$

(f) $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$

(g) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

17. The n^{th} Roots of Unity :

The n^{th} roots of unity, it means any complex numebr z, which satisfies the equation $z^n = 1$ or $z = (1)^{1/n}$ or $z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, where $k = 0, 1, 2, \ldots, (n-1)$

18. Properties of n^{th} Roots of Unity :

(i) n^{th} roots of unity form a GP with common ratio $e^{\frac{i2\pi}{n}}$.

(ii) Sum of n^{th} roots of unity is always 0.

(iii) Sum of p^{th} powers of n^{th} roots of unity is n, if p is a multiple of n.

(iv) Sum of p^{th} powers of n^{th} roots of unity is zero, if p is not a multiple of n.

(v) Product of n^{th} roots of unity is $(-1)^{n-1}$,

(vi) The n^{th} roots of unity lie on the unit circle $|z| = 1$ and divide its circumference into n equal parts.

19. Some Important Results

If $z = \cos\theta + i\sin\theta$

(i) $z + \frac{1}{z} = 2\cos\theta$

(ii) $z - \frac{1}{z} = 2i\sin\theta$

(iii) $z^n + \frac{1}{z^n} = 2\cos n\theta$, $z^n - \frac{1}{z^n} = 2i\sin n\theta$

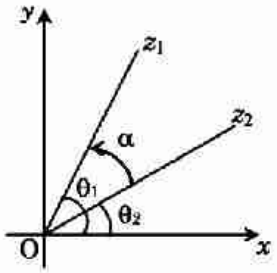
(iv) If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and given, $x + y + z = 0$, then

(a) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$ (b) $yz + zx + xy = 0$ (c) $x^2 + y^2 + z^2 = 0$ (d) $x^3 + y^3 + z^3 = 3xyz$

20. Concept of Rotation

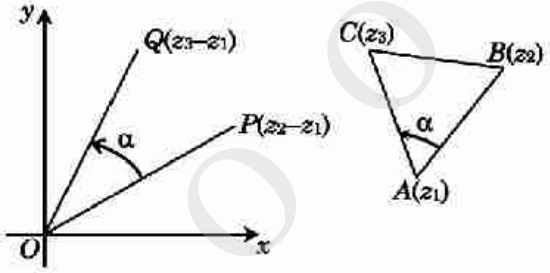
If z_1 and z_2 are two complex number then argument of

$$\frac{z_1}{z_2} = \frac{|z_1|e^{i\theta_1}}{|z_2|e^{i\theta_2}} = \left|\frac{z_1}{z_2}\right| e^{i(\theta_1-\theta_2)} = \left|\frac{z_1}{z_2}\right| e^{i\alpha}$$



In general, let z_1, z_2, z_3 be the three vertices of a ΔABC described in the counter clockwise sense. Draw OP and OQ parallel and equal to AB and AC respectively. Then P is $z_2 - z_1$ and Q is $z_3 - z_1$ then

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\alpha}$$



21. Application of Complex Number in Co-ordinate Geometry

(i) Distance between complex points :

(a) The distance between two points $P(z_1)$ and $Q(z_2)$ is given by

$PQ = |z_2 - z_1|$

(b) Section formula : If $R(z)$ divides the line segment joining $P(z_1)$ and $Q(z_2)$ in the ratio $m_1 : m_2$ (m_1, $m_2 > 0$). Then,

For internal division $z = \dfrac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$

For external division $z = \dfrac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$

(ii) Triangle in Complex Plane :

(a) If the vertices A, B, C of a Δ represent the complex nos. z_1, z_2, z_3 respectively, then

Centroid of the $\Delta ABC = \dfrac{z_1 + z_2 + z_3}{3}$

Incentre of the $\Delta ABC = \dfrac{az_1 + bz_2 + cz_3}{a + b + c}$

(b) Area of the triangle with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$$

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(c) The triangle whose vertices are the points represented by complex numbers z_1, z_2, z_3 is equilateral if $\frac{1}{z_1-z_2}+\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}=0$

or $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$.

(iii) **Straight line in Complex Plane :**

(a) General equation of a line in complex plane is $\bar{a}z+a\bar{z}+b=0$ where $b\in R$ and a is a fixed non zero complex number.

(b) The complex and real slope of the line $\bar{a}z+a\bar{z}=k$ are $\left(-\frac{a}{\bar{a}}\right)$ and $\left(-\frac{Re(a)}{Im(a)}\right)$.

(c) Equation of the line joining complex number z_1 and z_2 is

$$z=tz_1+(1-t)z_1,\ t\in R \text{ or } \frac{z-z_1}{z_2-z_1}=\frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}=\begin{vmatrix} z & \bar{z} & 1\\ z_1 & \bar{z}_1 & 1\\ z_2 & \bar{z}_2 & 1\end{vmatrix}=0$$

(d) z_1z_2 are two fixed points, then $|z-z_1|=|z-z_2|$ represents perpendicular bisector of the line segment joining z_1 and z_2.

(e) The length of perpendicular form a point z_1 to $\bar{a}z+a\bar{z}+b=0$ is given by $\frac{|\bar{a}z_1+a\bar{z}_1+b|}{2|a|}$

(f) The equation of a line parallel to the line $\bar{a}z+a\bar{z}+b=0$ is $\bar{a}z+a\bar{z}+\lambda=0$, where $\lambda\in R$.

(g) The equation of a line perpendicular to the line $\bar{a}z+a\bar{z}+b=0$ is $a\bar{z}-\bar{a}z+i\lambda=0$, where $\lambda\in R$.

(h) The equation of the perpendicular bisector of the line segment joining points A (z_1) and B (z_2) is $z(\bar{z}_1-\bar{z}_2)+\bar{z}(z_1-z_2)=|z_1|^2-|z_2|^2$

(iii) **Circle in Complex Plane :**

(a) Equation of circle with centre z_0 and radius r is $|z-z_0|=r$ or $z\bar{z}+z_0\bar{z}_0-z\bar{z}_0-\bar{z}z_0=r^2$

(b) General equation of a circle is $z\bar{z}+a\bar{z}+\bar{a}z+b=0$ where $b\in R$ and a is fixed complex number.

For this circle, centre is the points a and radius = $\sqrt{|a|^2-b}$

(c) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is, the number $\frac{z_3-z_1}{z_3-z_2}\cdot\frac{z_4-z_2}{z_4-z_1}$ is real.

(d) The equation of the circle described on the line segment joining z_1 & z_2 as diameter is $(z-z_1)(\bar{z}-\bar{z}_2)+(z-z_2)(\bar{z}-\bar{z}_1)=0$

22. Standard Loci

(i) $|z-z_1|+|z-z_2|=2k$ (a constant) represent

(a) If $2k>|z_1-z_2|\Rightarrow$ An ellipse (z_1 and z_2 are foci)

(b) If $2k=|z_1-z_2|\Rightarrow$ A line segment

(c) If $2k<|z_1-z_2|\Rightarrow$ No solution

(ii) Equation $||z-z_1|-z-z_2||=2k$ (a constant) represent

(a) If $2k<|z_1-z_2|\Rightarrow$ A Hyperbola (z_1 and z_2 are foci)

(b) If $2k=|z_1-z_2|\Rightarrow$ A line break

(c) If $2k>|z_1-z_2|\Rightarrow$ No solution

(iii) If $\frac{|z-z_1|}{|z-z_2|}=k$ ($k\neq 1$) then locus of z is a circle

(iv) If $|z-z_1|^2+|z-z_2|^2=|z_1-z_2|^2$ then locus of z is a circle with z_1, z_2 are the end points of diameter.

(v) If $\arg\left(\frac{z-z_1}{z-z_2}\right)=\pm\frac{\pi}{2}$ then locus of z is a circle with z_1, z_2 are the end points of diameter.

(vi) If $|z-z_1|^2+|z-z_2|^2=k$ where $\frac{1}{2}|z_1-z_2|^2\le k$ then locus of z is a circle.

SHORT TRICKS

1. Square root of Complex Number

Formula :

$$\sqrt{z} = \sqrt{x+iy} = \pm\left[\sqrt{\frac{|z|+x}{2}} + i\sqrt{\frac{|z|-x}{2}}\right]$$

$$\sqrt{z} = \sqrt{x-iy} = \pm\left[\sqrt{\frac{|z|+x}{2}} - i\sqrt{\frac{|z|-x}{2}}\right]$$

Trick -1 To find $\sqrt{x+iy}$ first find the number $\frac{|y|}{2}$ and then factorised this number in such a way that difference of square of these factors is equal to real number (x)

Q.1 Square root of $7 + 24i$ is -

(A) $\pm(3 - 4i)$ (B) $\pm(4 + 3i)$ (C) $\pm(4 - 3i)$ (D) $\pm(-3 - 4i)$

Sol. **[B]**

Proper Method

Let $\sqrt{7+24i} = a + ib$

$7 + 24i = (a + ib)^2$

$7 + 24i = a^2 - b^2 + 2iab$

$\Rightarrow a^2 - b^2 = 7$ ...(1)

and $2ab = 24$ ...(2)

$a^2 + b^2 = \pm\sqrt{(a^2-b^2)^2 + 4a^2b^2}$

$= \pm\sqrt{49+576}$

$a^2 + b^2 = \pm 25$ ...(3)

Solving (1) & (3)

$2a^2 = 7 \pm 25$

$a^2 = 16 \Rightarrow a \pm 4$

and $b = \pm 3$

$\therefore \sqrt{7+24i} = \pm(4 + 3i)$

Short Trick

$\sqrt{7+24i}$

↓

12

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4 3

Ans. $= \pm(4 + 3i)$

Q.2 Square root of $-5 - 12i$ is -

(A) $\pm(-2 - 3i)$ (B) $\pm(3 - 2i)$ (C) $\pm(2 - 3i)$ (D) $\pm(-3 - 2i)$

Sol. **[C]**

Proper Method

$\sqrt{-5-12i} = a + ib$

$-5 - 12i = (a^2 - b^2) + 2iab$

$\Rightarrow a^2 - b^2 = -5$ ...(1)

and $2ab = -12$ ...(2)

$a^2 + b^2 = \pm\sqrt{(a^2-b^2)^2 + 4a^2b^2}$

$a^2 + b^2 = \pm\sqrt{25+144}$

$a^2 + b^2 = \pm 13$ ...(3)

Short Trick

$\sqrt{-5-12i}$

↓

6

↙ ↘

2 3

Ans. $= \pm(2 - 3i)$

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Solving (1) & (3) $\Rightarrow 2a^2 = -5 \pm 13$ $\Rightarrow a^2 = 4$ $\Rightarrow \boxed{a = \pm 2}$ and $b = \mp 3$ $\therefore \sqrt{5-12i} = \pm(2-3i)$	

2. Application of triangle inequality to find minimum or maximum value

(i) $|z_1 \pm z_2| \geq \big||z_1| - |z_2|\big|$

(ii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

Trick -2 If $|z + z_1| \leq x$ (Real number) then to find maximum and minimum value of $|z + z_2|$ we write $|z + z_2|$ such that $|z + z_2| = |(z + z_1) \pm z_3|$

Now, max. value of $|z + z_2| = |x| + |z_3|$

min. value of $|z + z_2| = ||x| - |z_3||$

Q.3 If $|z + 4| \leq 3$ then maximum and minimum value of $|z + 1|$ is -

(A) 4, 0 (B) 10, 2 (C) 6, 0 (D) 4, 2

Sol. **[C]**

Proper Method	**Short Trick**
$\|z+1\| = \|(z+4)-3\|$ $\|(z+4)-3\| \leq \|z+4\| + \|3\|$ $\leq 3 + \|3\|$ ≤ 6 and $\|(z+4)-3\| \geq \|z+4\| - \|3\|$ $\geq 3 - \|3\|$ ≥ 0	$\|z+1\| = \|(z+4)-3\|$ $\|z+1\|_{max} = \|3\| + \|-3\| = 6$ $\|z+1\|_{min} = \|3\| - \|-3\| = 0$

Q.4 If $|z + 4i| \leq 3$ then maximum and minimum value of $|z + 3|$ is -

(A) 8, 2 (B) 6, 2 (C) 8, 4 (D) 8, 0

Sol. **[A]**

Proper Method	**Short Trick**
$\|z+3\| = \|(z+4i)+(3-4i)\|$ $\|z+3\| \leq \|z+4i\| + \|3-4i\|$ $\leq 3 + 5$ ≤ 8 and $\|z+3\| \geq \big\|\|z+4i\| - \|3-4i\|\big\|$ $\geq \|3-5\|$ ≥ 2	$\|z+3\| = \|(z+4i)+(3-4i)\|$ $\|z+3\|_{max} = 3 + \|3-4i\| = 8$ $\|z+3\|_{min} = \|3 - \|3-4i\|\| = 2$

Q.5 If $|z - 3 + 2i| \leq 4$ then the difference between the greatest and least value of $|z|$ is –

(A) $\sqrt{13}$ (B) $2\sqrt{13}$ (C) 8 (D) $4 + \sqrt{13}$

Sol. **[B]**

Proper Method	**Short Trick**
$\|z\| = \|(z-3+2i)+(3-2i)\|$ $\|z\| \leq \|z-3+2i\| + \|3-2i\|$ $\|z\| \leq 4 + \sqrt{13}$	$\|z\| = \|z-3+2i+(3-2i)\|$ $\|z\|_{max} = 4 + \sqrt{13}$

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and $\|z\| \geq \|z-3+2i\| - \|3-2i\|$ $\|z\| \geq 4-\sqrt{13}$ Difference between the greater and least value of $\|z\| = (4+\sqrt{13}) - (4-\sqrt{13})$ $= 2\sqrt{13}$	$\|z\|_{min} = 4-\sqrt{13}$ Difference $= 2\sqrt{13}$

Trick -3 If $\left|z+\frac{1}{z}\right| = a$, then the greatest and least value of $|z|$ are respectively $\frac{a+\sqrt{a^2+4}}{2}$ and $\frac{-a+\sqrt{a^2+4}}{2}$

Q.6 If $\left|z-\frac{1}{z}\right| = 1$, then:

(A) $|z|_{max} = \frac{1+\sqrt{5}}{2}$ (B) $|z|_{min} = \frac{1+\sqrt{5}}{2}$

(C) $|z|_{max} = \frac{-1+\sqrt{5}}{2}$ (D) None of these

Sol. **[A]**

Proper Method	**Short Trick**
$\left\|z-\frac{1}{z}\right\| = 1 \Rightarrow \left\|\|z\|-\frac{1}{\|z\|}\right\| \leq 1$ $\Rightarrow -1 \leq \|z\| - 1/\|z\| \leq 1 \Rightarrow -\|z\| \leq \|z\|^2 - 1 \leq \|z\|$ $\therefore \|z\|^2 + \|z\| - 1 \geq 0$ and $\|z\|^2 - \|z\| - 1 \leq 0$ $\Rightarrow \frac{\sqrt{5}-1}{2} \leq \|z\| \leq \frac{\sqrt{5}+1}{2}$ $\therefore$ max. $\|z\| = \frac{\sqrt{5}+1}{2}$ and min. $\|z\| = \frac{\sqrt{5}-1}{2}$	$\|z\|_{max} = \frac{1+\sqrt{5}}{2}$

Trick -4 **Method of substitution.**

Q.7 If $z = x - iy$ and $z^{1/3} = p + iq$, then the value of $\frac{\left(\frac{x}{p}+\frac{y}{q}\right)}{(p^2+q^2)}$ is equal to :

(A) 2 (B) – 1 (C) 1 (D) – 2

Sol. **[D]**

Proper Method	**Short Trick**
$z = x - iy$ and $z^{1/3} = p + iq$ So, $z = (p+iq)^3$ $\Rightarrow x - iy = p^3 - iq^3 + 3p(iq)(p+iq)$ $\Rightarrow x - iy = p^3 - iq^3 + 3ip^2q - 3pq^2$ Comparing real and imaginary part $\Rightarrow x = p^3 - 3pq^2$ and $y = q^3 - 3p^2q$ $\Rightarrow x = p(p^2 - 3q^2)$ and $y = q(q^2 - 3p^2)$	Let $x = 1$, $y = 0$ then $z = 1$ and $z^{1/3} = 1^{1/3} = \omega$ (let) $\frac{-1}{2} + i\frac{\sqrt{3}}{2} = p + iq$ $\Rightarrow p = -\frac{1}{2}$ & $q = \frac{\sqrt{3}}{2}$

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$\Rightarrow \frac{x}{p} = p^2 - 3q^2$ ...(1) and $\frac{x}{q} = q^2 - 3p^2$ ...(2) adding (1) & (2) $\Rightarrow \frac{x}{p} = \frac{x}{q} = -2(p^2 + q^2)$ $\Rightarrow \frac{\frac{x}{p} + \frac{y}{q}}{p^2 + q^2} = -2$	so $\frac{x}{p} + \frac{y}{q} + (p^2 + q^2) = -2$

Q.8 Let z and w be non-zero complex numbers such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$. Then z =
(A) w (B) $-w$ (C) $\overline{w}$ (D) $-\overline{w}$

Sol. **[D]**

Proper Method	**Short Trick**
Let $\arg(z) = \theta$ then $\arg(w) = \pi - \theta$ $z = \|z\|(\cos\theta + i\sin\theta)$ $w = \|w\|\cos\{(\pi - \theta) + i\sin(\pi - \theta)\}$ $w = \|w\|(-\cos\theta + i\sin\theta)$ $w = -\|w\|(\cos\theta - i\sin\theta)$ $w = -\|z\|(\cos\theta - i\sin\theta)$ $w = -\bar{z}$ or $z = -\overline{w}$	$\therefore$ Let $z = 1 + i \Rightarrow \arg(z) = \frac{\pi}{4}$ and $w = -1 + i \Rightarrow \arg(w) = \frac{3\pi}{4}$. $\therefore z = 1 + i = -(-1 - i) = -\overline{w}$

Q.9 If $|z| = 1$ and $z \neq \pm 1$ then all the values of $\frac{z}{1 - z^2}$ lies on:

(A) A line which does not pass through origin. (B) $|z| = \sqrt{2}$
(C) x–axis (D) y–axis

Sol. **[D]**

Proper Method	**Short Trick**
Let $z = x + iy$ $\frac{z}{1 - z^2} = \frac{x + iy}{1 - (x + iy)^2}$ $= \frac{x + iy}{1 - (x^2 - y^2 + 2ixy)}$ $= \frac{x + iy}{(1 - x^2 + y^2 - 2ixy)}$ $= \frac{x + iy}{(1 - x^2 + y^2) - 2ixy} \times \frac{(1 - x^2 + y^2) + 2ixy}{(1 - x^2 + y^2) - 2ixy}$ $= \frac{x(1 - x^2 + y^2) + 2ix^2y + iy(1 - x^2 + y^2) - 2xy^2}{(1 - x^2 + y^2) + 4x^2y^2}$ $= \frac{x(1 - x^2 + y^2 - 2y^2) + iy(2x^2 + 1x^2 + y^2)}{(1 - x^2 + y^2) + 4x^2y^2}$	$\because \|z\| = 1$ therefore let $z = i$ then $\frac{z}{1 - z^2} = \frac{i}{1 - i^2} = \frac{i}{2}$ (purely imaginary) so all the values of given expression lie on y–axis.

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$= \frac{x(1-(x^2+y^2))+iy(1+x^2+y^2)}{(1-x^2+y^2)+4x^2y^2}$ Given that $\|z\| = 1 \Rightarrow x^2 + y^2 = 1$ $= \frac{2iy}{(1-x^2+y^2)^2+4x^2y^2}$ purely imaginary $\therefore$ lies on y-axis	

Q.10 If α and β are different complex numbers with $|\beta| = 1$, then $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$ is equal to -

(A) 0 (B) 1/2 (C) 1 (D) 2

Sol. **[C]**

Proper Method	**Short Trick**
$\|\beta\| = 1 \Rightarrow \bar{\beta} = \frac{1}{\beta}$ $\Rightarrow \frac{\|\beta-\alpha\|}{\|1-\bar{\alpha}\beta\|} = \frac{\|\beta-\alpha\|}{\left\|1-\frac{\bar{\alpha}}{\bar{\beta}}\right\|}$ $= \|\bar{\beta}\| \cdot \frac{\|\beta-\alpha\|}{\|\bar{\beta}-\bar{\alpha}\|} = \|\bar{\beta}\| = 1$	Let $\alpha = i$ and $\beta = -i$ $\left\|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right\| = \left\|\frac{-i-i}{1-i^2}\right\| = 1$

Q.11 If $|z| = 1$, $z \neq -1$ and $w = \frac{z-1}{z+1}$ then real part of w = ?

(A) $\frac{-1}{|z+1|^2}$ (B) $\frac{1}{|z+1|^2}$ (C) $\frac{2}{|z+1|^2}$ (D) 0

Sol. **[D]**

Proper Method	**Short Trick**
$\|z\| = 1 \Rightarrow \bar{z} = \frac{1}{z}$ $w = \frac{z-1}{z+1}$ $\bar{w} = \frac{\bar{z}-1}{\bar{z}+1} = \frac{\frac{1}{z}-1}{\frac{1}{z}+1} \Rightarrow \bar{w} = \frac{1-z}{1+z}$ $\therefore w + \bar{w} = 0$ $\therefore$ Re(w) = 0	Let $z = i$ Then $w = \frac{i-1}{i+1} \times \frac{i-1}{i-1} = i$ $\therefore$ Re(w) = 0

Q.12 Let $z_1, z_2, \ldots, z_n$ be in G.P. with first term as unity such that $z_1 + z_2 + \ldots + z_n = 0$. Now if $z_1, z_2, \ldots z_n$ represents vertices of a polygon, then distance between the incentre and circumcentre of the polygon is :

(A) 2 (B) 1 (C) 0 (D) –1

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Sol. [C]

Proper Method	Short Trick
Let r be the common ratio of the G.P. $z_1, z_2, \ldots, z_n$ with $z_1 = 1$. $z_1 + z_2 + \ldots + z_n = 0 \Rightarrow \frac{(r^n - 1)}{r-1} = 0\ (r \neq 1)$ $\Rightarrow r^n = 1 = \lvert r \rvert = 1$ $\Rightarrow \lvert z_k \rvert = \lvert r^k \rvert = \lvert r \rvert^k = 1$ $\Rightarrow$ The polygon $z_1, z_2, \ldots, z_n$ is regular $\Rightarrow$ incentre and circumcentre are same $\Rightarrow$ distance between incentre and circumcentre is zero	Let $z_1 = 1, z_2 = \omega, z_3 = \omega^2$ Now z_1, z_2, z_3 are the vertices of equilateral triangle then distance between incentre and circumcentre is 0

Trick -5 Combination of method of substitution and balancing.

Q.13 If ω is a imaginary roots of unity then the value of following series
$(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) + 3(4-\omega)(4-\omega^2) + \ldots\ldots\ldots + (n-1)(n-\omega)(n-\omega^2)$ is :

(A) $\frac{n}{4}(n-1)(n^2+3n+4)$ (B) $\frac{n}{2}(n-1)(n^2+3n+4)$

(C) $\frac{n}{4}(n+1)(n^2+3n+4)$ (D) $\frac{n}{2}(n+1)(n^2+3n+4)$

Sol. [A]

Proper Method	Short Trick
$\text{sum} = \sum_{r=1}^{n-1} r(r+1-\omega)(r+1-\omega^2)$ $= \sum_{r-1}^{n-1} r[(r+1)^2 - (r+1)(\omega+\omega^2) + \omega^3]$ $= \sum_{r=1}^{n-1} (r^3 + 3r^2 + 3r)$ $= \sum_{r=1}^{n-1} r^3 + 3\sum_{r=1}^{n-1} r^2 + 3\sum_{r=1}^{n-1} r$ $= \left[\frac{1}{2}n(n-1)\right]^2 + 3\,\frac{(n-1)(n)(2n-1)}{6} + \frac{3}{2}(n-1)(n)$ $= \frac{n(n-1)}{2}\left(\frac{n(n-1)}{2} + (2n-1) + 3\right)$ $= \frac{n(n-1)}{2}\left\{\frac{n^2-n+4n-2+6}{2}\right\}$ $= \frac{n(n-1)}{4}(n^2+3n+4)$	Let $n = 2$, L.H.S $= (2-\omega)(2-\omega^2)$ $= 4 - 2(\omega^2+\omega) + \omega^3 = 4 - 2(-1) + 1 = 7$ Now put $n = 2$ in options then option (A) will give $\frac{2}{4}(2-1)(2^2+3.2+4) = 7$

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Q.14 The set of all $\alpha \in R$, for which $w = \frac{1+(1-8\alpha)z}{1-z}$ is a purely imaginary number, for all $z \in C$ satisfying $|z| = 1$ and $Re(z) \neq 1$, is

(A) {0} (B) an empty set (C) $\left\{0, \frac{1}{4}, -\frac{1}{4}\right\}$ (D) equal to R

Sol. **[A]**

Proper Method	**Short Trick**
$w = \frac{1+(1-8\alpha)z}{1-z}$ $w = \frac{1+(1-8\alpha)(x+iy)}{1-(x+iy)}$ $w = \frac{1+x(1-8\alpha)+iy(1-8\alpha)}{(1-x)-iy} \times \frac{(1-x)+iy}{(1-x)+iy}$ $w = \frac{\{1+x(1-8\alpha)\}(1-x)-y^2(1-8\alpha) + iy\{1+x(1-8\alpha)\}+iy(1-8\alpha)(1-x)}{(1-x)^2+y^2}$ given that w is purely imaginary $\therefore Re(w) = 0$ $\Rightarrow \{1+x(1-8\alpha)\}(1-x)-y^2(1-8\alpha) = 0$ $\Rightarrow 1-x+x(1-8\alpha)-x^2(1-8\alpha)-y^2(1-8\alpha) = 0$ $\Rightarrow 1-x+x-8\alpha x-(1-8\alpha)(x^2+y^2) = 0$ [given that $\|z\| = 1 \therefore x^2+y^2 = 1$] $\Rightarrow 1-8\alpha x-1+8\alpha = 0$ $\Rightarrow 8\alpha(-x+1) = 0$ $\Rightarrow 8\alpha = 0 \quad (\because x \neq 1)$ $\Rightarrow \alpha = 0$	$\|z\| = 1$ and $Re(z) \neq 1$ So assume $z = i$ $w = \frac{1+(1-8\alpha)i}{1-i} \times \frac{1+i}{1+i}$ $= \frac{(1-1+8\alpha)+i(1+1-8\alpha)}{2}$ w is purely imaginary $\therefore Re(w) = 0$ $\Rightarrow 8\alpha = 0 \Rightarrow \alpha = 0$

Q.15 If $c^2 + s^2 = 1$, then $\frac{1+c+is}{1+c-is} =$

(A) c + is (B) s + ic (C) c – is (D) s – ic

Sol. **[A]**

Proper Method	**Short Trick**
$\because c^2 + s^2 = 1$ $\therefore (c+is)(c-is) = 1$ $\therefore c-is = \frac{1}{c+is}$ $\therefore \frac{1+(c+is)}{1+(c-is)} = \frac{1+(c+is)}{1+\left(\frac{1}{c+is}\right)}$ $= (c+is)$	Let $c + is = i$ $(\because c^2 + s^2 = 1)$ $\frac{1+c+is}{1+c-is} = \frac{1+i}{1-i} = i$

Q.16 If $x = a + b$, $y = a\omega + b\omega^2$, $z = a\omega^2 + b\omega$, then xyz equals-

(A) $(a+b)^3$ (B) $a^3 - b^3$ (C) $(a+b)^3 + 3ab(a+b)$ (D) $a^3 + b^3$

Sol. **[D]**

Proper Method	**Short Trick**
$xyz = (a+b)(a\omega+b\omega^2)(a\omega^2+b\omega)$ $= (a+b)[a^2+b^2+ab(\omega^2+\omega)]$ $= (a+b)(a^2+b^2-ab) = a^3+b^3$	Let $a = 1$ and $b = 1$ Then $x = 2$, $b = \omega + \omega^2 = -1$, $c = \omega^2 + \omega = -1$ Now xyz = 2

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Q.17 If $|z_1| = |z_2| .... = |z_n| = 1$, then $\left|\frac{z_1 + z_2 + ... + z_n}{z_1^{-1} + z_2^{-1} + ... + z_n^{-1}}\right|$ equals-

(A) 1/n (B) n (C) 1 (D) $|z_1 + z_2 + ...+ z_n|$

Sol. **[C]**

Proper Method	**Short Trick**
$\|z_1\| = \|z_2\| ..... = \|z_n\| = 1$ $\therefore \bar{z}_1 = \frac{1}{z_1}, \bar{z}_2 = \frac{1}{z_2} .....$ $= \frac{\|z_1 + z_2 ..... + z_n\|}{\|\bar{z}_1 + \bar{z}_2 + ... + \bar{z}_n\|}$ = 1 (they are conjugate)	Let n = 2, then $\|z_1\| = \|z_2\| = 1$ also $z = 1$ and $z_2 = i$ $\left\|\frac{z_1 + z_2}{z_1^{-1} + z_2^{-1}}\right\| = \left\|\frac{1+i}{1-i}\right\| = 1$

Q.18 The value of the expression

$\left(1+\frac{1}{\omega}\right)\left(1+\frac{1}{\omega^2}\right) + \left(2+\frac{1}{\omega}\right)\left(2+\frac{1}{\omega^2}\right) + \left(3+\frac{1}{\omega}\right)\left(3+\frac{1}{\omega^2}\right) + ... + \left(n+\frac{1}{\omega}\right)\left(n+\frac{1}{\omega^2}\right)$, where ω is an imaginary cube root of unity is-

(A) $\frac{n(n^2+3)}{3}$ (B) $\frac{n(n^2+2)}{3}$ (C) $\frac{n(n^2+1)}{3}$ (D) None of these

Sol. **[B]**

Proper Method	**Short Trick**
$T_r = \left(r+\frac{1}{\omega}\right)\left(r+\frac{1}{\omega^2}\right) = (r+\omega^2)(r+w)$ $\therefore T_r = r^2 + 1 - r$ $\therefore \sum_{r=1}^{n} T_r = \Sigma r^2 - \Sigma r + \Sigma 1$ $= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n$ $= \frac{n}{6}[2n^2 + 3n + 1 - 3n - 3 + 6]$ $= \frac{n}{6}(2n^2 + 4)$ $= \frac{n}{3}(n^2 + 2)$	Put n = 1 $\Rightarrow \left(1+\frac{1}{\omega}\right)\left(1+\frac{1}{\omega^2}\right)$ $= (1 + \omega^2)(1 + \omega)$ $= (-\omega)(-\omega^2) = 1$ Now put n = 1 in option then option (B) $\frac{n(n^2+2)}{3}$ is the correct answer.

Q.19 If 1, a_1, a_2, a_3, ... a_{n-1} are n^{th} roots of unity then $\frac{1}{1-a_1} + \frac{1}{1-a_2} + ... + \frac{1}{1-a_{n-1}}$ equal to -

(A) $\frac{n-1}{2}$ (B) $\frac{n}{2}$ (C) $\frac{2^n - 1}{n}$ (D) none

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Sol. **[A]**

Proper Method

$z = (1)^{1/n}$

$z^n - 1 = 0$ have roots $1, a_1, a_2, \ldots, a_{n-1}$

Let $z = \dfrac{1}{1-z}$

$\dfrac{1}{z} = 1 - z$

$z = 1 - \dfrac{1}{z} = \dfrac{z-1}{z}$

Now replace z with $\dfrac{z-1}{z}$ equation becomes

$\left(\dfrac{z-1}{z}\right)^n - 1 = 0$

$(z-1)^n - z^n = 0$

$-{}^nC_1(z)^{n-1} + {}^nC_2(z)^{n-2} \ldots = 0$

$\therefore$ sum of roots

$= \dfrac{1}{1-a_1} + \dfrac{1}{1-a_2} + \ldots + \dfrac{1}{1-a_{n-1}}$

$= (-1)^1 \dfrac{{}^nC_2}{-{}^nC_1} = \dfrac{n(n-1)/2!}{n}$

$= \dfrac{n-1}{2}$

Short Trick

Let n = 3 then $a_1 = \omega$, $a_2 = \omega^2$

$\dfrac{1}{1-a_1} + \dfrac{1}{1-a_2} = \dfrac{1}{1-\omega} + \dfrac{1}{1-\omega^2}$

$= \dfrac{1-\omega^2+1-\omega}{(1-\omega^2-\omega+\omega^3)} = \dfrac{2-(\omega+\omega^2)}{2-(\omega+\omega^2)} = 1$

Now, put n = 3 in option, then option (A) $\dfrac{n-1}{2}$ is the correct answer.

Q.20 $1, z_1, z_2, z_3, \ldots z_{n-1}$ are nth roots of unity then the value of $\dfrac{1}{3-z_1} + \dfrac{1}{3-z_2} + \ldots + \dfrac{1}{3-z_{n-1}}$ is equal to:

(A) $\dfrac{n3^{n-1}}{3^n-1} + \dfrac{1}{2}$ (B) $\dfrac{n3^{n-1}}{3^n-1} - 1$ (C) $\dfrac{n3^{n-1}}{3^n-1} + 1$ (D) None of these

Sol. **[D]**

Proper Method

$1, z_1, z_2, \ldots, z_n$ n, nth roots of unity

$\Rightarrow z^n - 1 = (z-1)(z-z_1)(z-z_2)\ldots(z-z_{n-1})$

$\Rightarrow \log(z^n-1) = \log(z-1) + \log(z-z_2) + \ldots + \log(z-z_{n-1})$

Differentiating both sides, we get

$\dfrac{nz^{n-1}}{z^n-1} = \dfrac{1}{z-1} + \dfrac{1}{z-z_1} + \dfrac{1}{z-z_2} + \ldots + \dfrac{1}{z-z_{n-1}}$

Putting z = 3, we get

$\dfrac{n3^{n-1}}{3^n-1} = \dfrac{1}{2} + \dfrac{1}{3-z_1} + \dfrac{1}{3-z_2} + \ldots + \dfrac{1}{3-z_{n-1}}$

$\Rightarrow \dfrac{1}{3-z_1} + \dfrac{1}{3-z_2} + \ldots + \dfrac{1}{3-z_{n-1}} = \dfrac{n3^{n-1}}{3^n-1} - \dfrac{1}{2}$

Short Trick

Let n = 3, then $z_1 = \omega$, $z_2 = \omega^2$

Now $= \dfrac{1}{3-\omega} + \dfrac{1}{3-\omega^2}$

$= \dfrac{3-\omega^2+3-\omega}{9-3(\omega+\omega^2)+\omega^3} = \dfrac{7}{13}$

in option we put n = 3 then (D) is correct

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